Name:________________________________ Date:___________________

Vieta Sum Extra Challenge Problems

1. Three of the roots of $x^4 + ax^2 + bx + c = 0$ are 2, -3, and 5. Find the values of "a", "b" and "c"

2. Find the sum of all the roots of the equation: $x^{2001} + \left(0.5 - x\right)^{2001} = 0$ (aime 2001)

3. Suppose the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are "a", "b", and "c", and the roots of $x^3 + rx^2 + sx + t = 0$ are "a+b", "b+c" and "c+a", find the values of "r" and "t" {aime1996}. Finding "t' is much harder than finding "r"

4. Let "m', "n" and "p" be the roots of $x^3 - 3x^2 + 1$,
   a. Find a polynomial whose roots are "m+3, n+3, and p+3"

   b. What is the value of $\frac{1}{m+3} + \frac{1}{n+3} + \frac{1}{p+3}$

   c. Find a polynomial whose roots are $m^2,\ n^2,\ and\ p^2$

5. Solve the systems of equations:

$i)\ x+y+z=17$
$xy+yz+xz=94$
$xyz=168$

$ii)\ x+y-z=0$
$xz-xy+yz=27$
$xyz=54$

6. The product of two of the four zeros of the quartic equation below is equal to -32. Find the value of "k"
$x^4-18x^3+kx^2+200x-1984=0$

7. For some integer "a", the equations $1988x^2+ax+8891=0$ and $8891x^2+ax+1988=0$ share a common root. Find the value of "a" [Cad Olymp]

8. If P(x) is a polynomial in "x" such that for all "x",
$x^{23}+23x^{17}-18x^{16}-24x^{15}+108x^{14}=\left(x^4-3x^2-2x+9\right)\cdot P\left(x\right)$
Find the sum of the coefficients of $P\left(x\right)$

SOL:

The coefficient of x^3 is 0, so the sum of the roots is 0, and the fourth root must be -4. The polynomial factors as $(x-2)(x-5)(x+3)(x+4)$. Setting $x = 1$ gives $1 + a + b + c = (1-2)(1-5)(1+3)(1+4) = 80$, so $a + b + c = 79$.

Using Vieta's formulas is left as an exercise. A shortcut solution is to observe that if x is a root, so is $\frac{1}{2} - x$, and each such pair yields a total of $\frac{1}{2}$. There are 2000 roots, so the total is 500.

- We have $r = -((a+b)+(b+c)+(a+c)) =$ $= -2(a+b+c) = 6$.
- To evaluate $t = -(a+b)(a+c)(b+c)$, it helps to observe $a + b + c = -3$ and write $a + b = -3 - c$ and so on. Then $t = -P(-3) = 23$.

To increment all roots by 3, substitute $x - 3$ for x. This yields $x^3 - 12x^2 + 45x - 53$.

Reversing the coefficients to get $1 - 12x + 45x^2 - 53x^3$ yields a polynomial whose roots are reciprocals of the polynomial above. (Do you see why?)

This is just the sum of the roots of the polynomial above: $\frac{45}{53}$.

Observe that $x^3 + 3x^2 - 1$ has roots $-\alpha$, $-\beta$, and $-\gamma$. Therefore $(x^3 - 3x^2 + 1)(x^3 + 3x^2 - 1)$ has roots $\pm\alpha, \pm\beta, \pm\gamma$ and factors as $(x^2 - \alpha^2)(x^2 - \beta^2)(x^2 - \gamma^2)$. Replacing x^2 by x yields our answer: $x^3 - 9x^2 + 6x - 1$.

The recurrence is $x_n = 3x_{n-1} - x_{n-3}$. We have $x_0 = 3$, $x_1 = 3$, and $x_2 = 9$, so $x_3 = 24$, $x_4 = 69$, and $x_5 = 198$.

We start by trying to utilize the symmetry in the equations, and we note that each line of equations is a symmetric sum. Thus, we are given that $\sigma_1 = 17, \sigma_2 = 94, \sigma_3 = 168$.

Using Vieta's formulas, we are able to connect the notion of symmetric sums and roots of a polynomial. Now, using the results established in the previous post, we see that are the roots of the polynomial

$$f(a) = a^3 - 17a^2 + 94a - 168.$$

If we can find the roots of this polynomial, we can solve the system of equations, and we are done. Using the Rational Root Theorem and the Factor Theorem (explained perhaps some other day), we find that 4 is a root of $f(a)$, and after we factor out the $(a-4)$, we have a quadratic, and by quickly factoring the quadratic, we have that

$$f(a) = (a-4)(a-6)(a-7).$$

Thus, we have that the solution triples of (a, b, c) are $\boxed{(4,6,7),(4,7,6),(6,4,7),(6,4,7),(7,4,6),(7,6,4)}$.

$$
\begin{aligned}
&a+b+c+d=18 &&...(1)\\
&ab+bc+cd+da+ac+bd=k &&...(2)\\
&abc+acd+abd+bcd=-200 &&...(3)\\
&abcd=1984 &&...(4)
\end{aligned}
$$

We let the two roots that multiply to 32 be a and b, so we have $ab=-32$. Immediately, we have $cd=62$ Now, we factor (3) and substitute these values:

$$ab(c+d)+cd(a+b)=-200.$$

So then we have,

$$-32(c+d)+62(a+b)=-200.$$

If we let $\alpha=a+b$ and $\beta=c+d$, we have the relations

$$\alpha+\beta=18$$

$$-32\alpha+62\beta=-200$$

Solving this system of equations, we have that a + b =14 and c + d = 4. Finally, substituting these values into (2), we have:

k = ab + (a + b)(c+d) + cd = -32 + (14)(4) + 62.

Thus, the answer is $\boxed{k=86}$

Let x be the common root; then by subtracting the two equations, we have

$$(8891-1988)x^2+(1988-8891)=0$$

so $x^2-1=0$, and therefore $x=\pm1$. Plug ±1 into one of the equations to get $1988\pm a+8891=0$ and therefore $a=\pm10879$.

The sum of the coefficients of $P(x)$ is $P(1)$. Setting $x=1$, we get

$$1+23-18-24+108=(1-3-2+9)P(1).$$

Solving, we obtain $P(1)=18$.